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# NOTES

# Spurious Modes in Spectral Collocation Methods with Two Non-periodic Directions

## 1. INTRODUCTION

Most existing numerical solutions of incompressible Navier–Stokes equations in three dimensions employ the primitive-variable formulation. Here, the velocity field and pressure cannot be approximated independently and must satisfy a compatibility condition. Approximating both velocity and pressure by polynomials of the same degree will result in some spurious modes for the pressure [1]. In a spectral collocation implementation, these spurious modes can be characterized as the spurious components of the pressure field whose discrete gradient at the interior collocation points, where discretized momentum equations are satisfied, is zero. Such pressure components are left uncontrolled by the discretized governing equations, whereas, these spurious pressure modes do not affect the velocity field, since pressure enters the momentum equation only through the pressure gradient term. Yet, this error committed in the pressure field can significantly affect quantities of interest, such as pressure fluctuation statistics, pressure–strain correlation, $C_p$ distribution, and net pressure forces.

In coupled spectral implementations where the continuity equation is discretized directly, theoretical analysis of the spurious modes is on a firm footing [1–6]. Earlier investigators have pointed out the advantages of a staggered mesh in eliminating the spurious modes [4–6]. On the other hand, the analysis of spurious modes in uncoupled spectral implementations, where a pressure Poisson equation is solved is not complete. Spurious modes in a one-dimensional implementation of the Kleiser–Schumann's influence matrix method has been discussed in [7]. Here we identify the spurious modes in the three-dimensional collocation implementation of the Kleiser–Schumann method [8–10] with two non-periodic directions. A simple correction procedure which will automatically filter these spurious modes is discussed. This correction procedure is applied in the simulation of a turbulent square duct flow [10] and is found to be very effective in eliminating all the spurious modes.

## 2. SPURIOUS MODES FOR KLEISER–SCHUMANN'S METHOD

The full implementation of the Kleiser–Schumann's method with *collocation* correction can be given by the equations and boundary conditions,

$$\nabla^2 p = -\nabla \cdot (\mathbf{NL}) + \nabla \cdot \mathbf{B} \qquad \text{in } \boldsymbol{\Omega} \tag{1a}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{NL} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{V} + \mathbf{B} \qquad \text{in } \boldsymbol{\Omega}, \partial\boldsymbol{\Omega} \tag{1b}$$

$$\mathbf{V} = \mathbf{V}_b \qquad \text{on } \partial\boldsymbol{\Omega} \tag{1c}$$

$$\nabla \cdot \mathbf{V} = 0 \qquad \text{on } \partial\boldsymbol{\Omega} \tag{1d}$$

$$\mathbf{B} = 0 \qquad \text{in } \boldsymbol{\Omega}, \tag{1e}$$

where $\mathbf{NL}$ is the nonlinear term in the Navier–Stokes equation and $\mathbf{V}_b$ is the velocity boundary condition. The above equations and boundary conditions are in their discretized form, therefore the symbols $\nabla$, $\nabla\cdot$, and $\nabla^2$ represent discrete gradient, divergence, and Laplacian operators, and $\boldsymbol{\Omega}$ and $\partial\boldsymbol{\Omega}$ represent interior and boundary collocation points. Since the momentum equation is satisfied only in the interior, $\mathbf{B}$ represents the boundary momentum residual and is nonzero only on the boundary. The above formulation identically satisfies all the velocity boundary conditions and also results in a divergence-free velocity field both in the interior and on the boundary.

By definition, each spurious mode is a valid solution to the discretized governing equations and appropriate boundary conditions. The spurious modes have a non-zero contribution to pressure but have no effect on velocity, therefore $\mathbf{V}_{\mathrm{sp}} = 0$, where subscript "sp" stands for the spurious mode. The spurious pressure components therefore satisfy

$$\nabla^2 p_{\mathrm{sp}} = \nabla \cdot \mathbf{B}_{\mathrm{sp}} \qquad \text{in } \boldsymbol{\Omega} \tag{2a}$$

$$0 = -\nabla p_{\mathrm{sp}} + \mathbf{B}_{\mathrm{sp}} \qquad \text{in } \boldsymbol{\Omega}, \partial\boldsymbol{\Omega} \tag{2b}$$

$$\mathbf{B}_{\mathrm{sp}} = 0 \qquad \text{in } \boldsymbol{\Omega}, \tag{2c}$$



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where $\mathbf{B}_{sp}$ is the corresponding spurious boundary momentum residual. Any non-trivial solution to the above linear equations represents a spurious mode, which when added to the true solution will still satisfy the discretized Navier–Stokes equations and boundary conditions.

Eight solutions to the above equations can be identified. First of which is the non-spurious solution, $p_{sp} = \text{constant}$ and $\mathbf{B}_{sp} = 0$, which indicates that pressure is evaluated only up to an arbitrary additive constant in incompressible flows. The first two spurious modes are the *line* and *column* modes,

$$P_{sp} = T_{Nx}(x),$$
$$B_{1sp} = \begin{Bmatrix} (\pm 1)^{Nx}\,(Nx)^2 & \text{at} \quad x = \pm 1 \\ 0 & \text{otherwise} \end{Bmatrix}, \qquad (3a)$$
$$B_{2sp} = B_{3sp} = 0,$$

and

$$P_{sp} = T_{Ny}(y),$$
$$B_{2sp} = \begin{Bmatrix} (\pm 1)^{Ny}\,(Ny)^2 & \text{at} \quad y = \pm 1 \\ 0 & \text{otherwise} \end{Bmatrix}, \qquad (3b)$$
$$B_{1sp} = B_{3sp} = 0.$$

Here $(Nx+1)$ and $(Ny+1)$ are the number of points along the non-periodic Chebyshev directions and the third direction is at most periodic. $B_1$, $B_2$, and $B_3$ are the three components of the boundary momentum residual and only the normal component is nonzero. $T_{Nx}$ and $T_{Ny}$ are Chebyshev polynomials of the highest degree along $x$ and $y$. The third spurious mode is the *checkerboard* mode, with

$$P_{sp} = T_{Nx}(x)\, T_{Ny}(y),$$
$$B_{1sp} = \begin{Bmatrix} (\pm 1)^{Nx}\,(Nx)^2\, T_{Ny}(y), & \text{at} \quad x = \pm 1 \\ 0 & \text{otherwise} \end{Bmatrix},$$
$$B_{2sp} = \begin{Bmatrix} (\pm 1)^{Ny}\,(Ny)^2\, T_{Nx}(x), & \text{at} \quad y = \pm 1 \\ 0 & \text{otherwise} \end{Bmatrix}, \qquad (4)$$
$$B_{3sp} = 0.$$

These three spurious modes have no variation in the periodic $z$ direction and therefore contaminate only the zeroth mode along the $z$ direction. The other four spurious modes are the *corner* modes and each of them can have arbitrary variation along the $z$ direction. For example, let $f_{1,1}(z)$ be the arbitrary variation along the $x = 1$, $y = 1$ corner. Then the *corner* mode corresponding to this corner can now be written as

$$P_{sp} = \begin{Bmatrix} f_{1,1}(z), & \text{at} \quad x = y = 1 \\ 0, & \text{otherwise} \end{Bmatrix},$$
$$B_{1sp} = \begin{Bmatrix} f_{1,1}(z)\, D_{Nx}(x), & \text{at} \quad y = +1 \\ 0, & \text{otherwise} \end{Bmatrix},$$
$$B_{2sp} = \begin{Bmatrix} f_{1,1}(z)\, D_{Ny}(y), & \text{at} \quad x = +1 \\ 0, & \text{otherwise} \end{Bmatrix},$$
$$B_{3sp} = \begin{cases} \dfrac{df_{1,1}}{dz}, & \text{at} \quad x = y = 1, \\ 0, & \text{otherwise}, \end{cases} \qquad (5)$$

where $D_{Nx}(x)$ is the discrete derivative of the polynomial which collocates to zero at all points except at $x = 1$. The *corner* modes for the other three corners can be written similarly. The *corner* spurious modes simply reflect the fact that in a collocation implementation the pressure along the four cornerlines never enter into the computation and therefore their values remain unspecified.

## 3. FILTERING PROCEDURE

Implementation of the Kleiser–Schumann method involves the construction of an influence matrix. The solution of the pressure Poisson equation (Eq. (1a)) requires the knowledge of pressure boundary conditions ($p_b$) and boundary momentum residuals ($\mathbf{B}_b$) at the $(2N_x + 2N_y - 4)$ points, excluding the corner points. These unknowns are evaluated by requiring that continuity and normal momentum equation (with the residual) are satisfied at the boundary points, excluding the corner points. This provides the necessary $(4N_x + 4N_y - 8)$ linear equations for the unknown quantities. These equations are cast into the following matrix form, $\mathbf{Ax} = \mathbf{R}$, where $\mathbf{A}$ is the influence matrix, $\mathbf{x}$ is the unknown vector of boundary pressure and normal momentum residual, and $\mathbf{R}$ is the right-hand side. In a three-dimensional problem, a Fourier transform along the periodic $z$ direction will result in one influence matrix for each Fourier wavenumber $k_z$. Invertibility of the influence matrix is closely related to the presence of spurious modes. In particular, the influence matrix corresponding to $k_z = 0$ suffers from the *constant*, *line*, *column*, and *checkerboard* modes and therefore has four zero eigenvalues. The eigenvectors corresponding to these eigenvalues are the corresponding spurious boundary pressure and normal component of the momentum residual.

As suggested by Tuckerman [9], the non-invertibility of the influence matrix can be easily overcome by constructing a related matrix $\mathbf{A}' = \mathbf{M}\boldsymbol{\lambda}'\mathbf{M}^{-1}$, where $\mathbf{M}$ and $\mathbf{M}^{-1}$ are the eigenvector matrix of the original influence matrix and its inverse, respectively, and $\boldsymbol{\lambda}'$ is a diagonal matrix with the eigenvalues along the diagonal and with the zero eigenvalues replaced by some non-zero constant. The influence matrix now becomes invertible, i.e., $\mathbf{A}'^{-1} = \mathbf{M}(\mathbf{1}/\boldsymbol{\lambda}')\mathbf{M}^{-1}$, and the resulting pressure and boundary momentum residuals yield a divergence-free flow field independent of the constant that replaces the zero eigenvalue. Let the $p$th eigenvalue of the original influence matrix be zero and be replaced by a constant $c_p$. Let the corresponding $p$th eigenvector be $M_{ip}$, which is a vector of boundary pressures and momentum residuals corresponding to a linear combination of the spurious modes. Contribution of this $p$th mode to the unknown vector $x_i$ is then $M_{ip} b_p / c_p$, where $b_p$ is the projection of the right-hand side along the eigenvectors, given by $M_{pj}^{-1} R_j$. Once simple way to filter the four spurious modes will then be to set the arbitrary constant $c_p$ to be infinity. In other words, in the evaluation of $\mathbf{A}'^{-1}$, one over the zero eigenvalue is simply replaced by zero.

This filtering procedure was implemented in the computation of turbulent flow in a square duct [10]. The high frequency oscillations present in the pressure field due to the spurious modes were completely removed by this implicit filtering procedure. A posteriori filtering of the spurious pressure modes also produces the same result but at the expense of the added cost of the explicit filtering procedure. The buit-in filtering also has the added advantage of automatically setting the mean pressure to be zero.

## 4. SPURIOUS MODES IN PARTIAL AND TIME-SPLIT METHODS

Spurious modes for the partial implementation of the Kleiser–Schumann's method [11] without the *collocation* correction can be analyzed in similar fashion. It can be easily seen that the only admissible solution for the above set of equations is the non-spurious *constant* mode and there are no spurious modes present. This is confirmed in the numerical simulation by observing that the influence matrix for the $k_z = 0$ mode has only one zero eigenvalue and the corresponding constant mean pressure can be set to zero by replacing the zero eigenvalue by infinity.

Spurious modes in the time split implementation will depend on the exact boundary conditions employed for the intermediate star-level velocities ($\mathbf{V}^*$) and the pressure Poisson equation. Following Streett and Hussaini [12], if we employ $[(2\nabla p(t) - \nabla p(t-\delta t)) \cdot \tau]$ as the boundary condition for the tangential components of the intermediate star-level velocity, zero penetration for the normal velocity component, and zero Neumann boundary condition for the pressure, then we have the following equations satisfied by the spurious modes:

$$\begin{aligned}
\mathbf{V}^*_{\rm sp} &= \Delta t\, \nabla p_{\rm sp} && \text{in } \boldsymbol{\Omega} \\
\nabla^2 \mathbf{V}^*_{\rm sp} &= \frac{2\,\mathrm{Re}}{\Delta t} \mathbf{V}^*_{\rm sp} && \text{in } \boldsymbol{\Omega} \\
\mathbf{V}^*_{\rm sp} \cdot \boldsymbol{\eta} &= \nabla p_{\rm sp} \cdot \boldsymbol{\eta} = 0 && \text{in } \partial\boldsymbol{\Omega} \\
\mathbf{V}^*_{\rm sp} \cdot \boldsymbol{\tau} &= (2\nabla p_{\rm sp} - \nabla p_{\rm sp}(t-\delta t)) \cdot \boldsymbol{\tau} && \text{in } \partial\boldsymbol{\Omega},
\end{aligned} \tag{6}$$

where $\boldsymbol{\eta}$ and $\boldsymbol{\tau}$ are direction normal and tangential to the boundary. The analysis of the spurious modes is more complicated and also depends on the initial tangential pressure gradients on the boundary. With careful choice of initial conditions, the no penetration and pure Neumann boundary conditions will guarantee no spurious components.

## 5. CONCLUSION

Collocation implementation of the Kleiser–Schumann method in geometries with two non-periodic directions have three spurious modes—*line*, *column*, and *checkerboard*—contaminating the computed pressure field. The *corner* spurious modes are also present but they do not affect evaluation of the pressure-related quantities. The three spurious modes can be easily filtered out by replacing the zero eigenvalues of the influence matrix with infinity before solving for the unknown boundary pressure and momentum residuals. Partial implementation of the Kleiser–Schumann method without *collocation* correction admits no spurious modes. Spurious modes can also be avoided in time-split-implementations.

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